

Double-Angle and Half-Angle Identities

Using the sum identities from the last section, we can derive more formulas called the double-angle identities:

$$\sin(2x) = \sin(x+x) = \sin x \cos x + \cos x \sin x = 2 \sin x \cos x$$

$$\cos(2x) = \cos(x+x) = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x = 1 - 2 \sin^2 x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) = 2 \cos^2 x - 1$$

$$\tan(2x) = \tan(x+x) = \frac{\tan x + \tan x}{1 - \tan x \tan x} = \frac{2 \tan x}{1 - \tan^2 x}$$

Example: Find $\sin \frac{2\pi}{3}$, $\cos \frac{2\pi}{3}$, and $\tan \frac{2\pi}{3}$ using double-angle identities.

Example: Use the double angle identities to verify that $\cos(3x) = \cos^3 x - 3 \cos x \sin^2 x$ is an identity.

We can use the double-angle identities to derive identities for $\sin(x/2)$, $\cos(x/2)$, and $\tan(x/2)$. We call these the half-angle identities.

To get identities for $\cos(x/2)$ and $\sin(x/2)$, we solve two of the equations for $\cos(2x)$ for $\sin x$ and $\cos x$.

$$2 \cos^2 x - 1 = \cos(2x)$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\cos x = \pm \sqrt{\frac{1 + \cos(2x)}{2}}$$

$$1 - 2 \sin^2 x = \cos(2x)$$

$$-2 \sin^2 x = \cos(2x) - 1$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\sin x = \pm \sqrt{\frac{1 - \cos(2x)}{2}}$$

Since these equations work for any value of x , they also work if we replace x by $(x/2)$:

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}$$

$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}$$

We can then use these formulas to derive formulas for $\tan \frac{x}{2}$:

$$\tan \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}} \quad \tan \frac{x}{2} = \frac{\sin x}{1 + \cos x} \quad \tan \frac{x}{2} = \frac{1 - \cos x}{\sin x}$$

Examples: Use the half-angle identities to find the exact values of $\sin \frac{\pi}{8}$, $\cos \frac{\pi}{8}$, and $\tan \frac{\pi}{8}$.

Example: Prove that $\sin^2 \left(\frac{x}{2} \right) \cos^2 \left(\frac{x}{2} \right) = \frac{\sin^2 x}{4}$ is an identity.

Example: Find $\sin \alpha$, $\cos \alpha$, and $\tan \alpha$ if $\cos(2\alpha) = -1/3$ and $\pi < 2\alpha < 3\pi/2$.

Example: Find $\sin \alpha$, $\cos \alpha$, and $\tan \alpha$ if $\sin(\alpha/2) = 4/5$ and $\pi/4 < \alpha/2 < \pi/2$.